## Summer Workshop in Mathematics (SWIM) Kannur University, May, 2025 Krishna Hanumanthu Tutorial problems

## May 5, 2025

1. Show the following.

- (a)  $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$  for all  $n \ge 0$ . (b)  $\sum_{i=0}^{n} 4^i = \frac{1}{3}(4^{n+1}-1)$  for all  $n \ge 0$ . (c)  $\sum_{i=0}^{n} 1/3^i = \frac{3}{2}(1-\frac{1}{3^{n+1}})$  for all  $n \ge 0$ . (d)  $2^n \le n!$  for all  $n \ge 4$ .
- 2. Let  $f: X \to Y$  be a function between two sets X, Y. Let  $A, B \subset X$  and  $C, D \subset Y$  be arbitrary subsets. Determine if the following statements are true or false.

(a) 
$$f(A \cap B) = f(A) \cap f(B)$$
.

- (b)  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$
- (c)  $f(A \cup B) = f(A) \cup f(B)$ .
- (d)  $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$
- 3. Let A, B, C be subsets of a set X. Show the following.
  - (a)  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$
  - (b)  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$
  - (c)  $(A \cup B)^c = A^c \cap B^c$ .
  - (d)  $(A \cap B)^c = A^c \cup B^c$ .
- 4. Show the set  $\mathbb{Z}$  of integers is countable by using the function  $f : \mathbb{N} \to \mathbb{Z}$  defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

## May 6, 2025

- 5. Let  $f: A \to B, g: B \to C$  be functions. Determine if the following statements are true or false.
  - (a) If f, g are injective, then so is  $g \circ f$ .

- (b) If  $g \circ f$  is injective, then so is f.
- (c) If  $g \circ f$  is injective, then so is g.
- (d) If f, g are surjective, then so is  $g \circ f$ .
- (e) If  $g \circ f$  is surjective, then so is f.
- (f) If  $g \circ f$  is surjective, then so is g.
- 6. If  $A_1, \ldots, A_k$  are countable, then show that the Cartesian product  $A_1 \times \ldots \times A_k$  is also countable.
- 7. Show that a countable union of countable sets is countable.
- 8. Which of the following are equivalence relations on the given sets A.
  - (a)  $A = \mathbb{N} : a \sim b$  if a b is odd.
  - (b)  $A = \mathbb{N} : a \sim b$  if a b is even.
  - (c)  $A = \mathbb{N} : a \sim b$  if  $-4 \leq a b \leq 4$ .
  - (d)  $A = \mathbb{R} : a \sim b$  if |a| = |b|.
  - (e) A is the set of all  $2 \times 2$  real matrices:  $A \sim B$  if there exists an invertible matrix  $P \in A$  such that  $A = PBP^{-1}$ .
- 9. Fix a positive integer n. Consider the equivalence relation on  $\mathbb{Z}$  defined by  $a \sim b$  if a b is divisible by n. Determine the equivalence classes for this relation.
- 10. Negate the following statements. P denotes the set of prime natural numbers.
  - (a)  $\exists r \in \mathbb{Q}$  such that  $r^2 = 2$ .
  - (b)  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } y^3 = x.$
  - (c)  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } y^2 = x \text{ or } x \leq 0.$
  - (d)  $\forall p \in P$ , if p > 2 then p is odd.
  - (e)  $\forall n \in \mathbb{N}$  such that  $n > 0 \exists p_1, \ldots, p_r \in P$  such such that  $n = p_1 \cdot \ldots \cdot p_r$ .
  - (f)  $\forall$  continuous functions  $f : \mathbb{R} \to \mathbb{R}$  and  $\forall a, b \in \mathbb{R}$  such that f(a) < f(b), if  $f(a) < c < f(b), \exists x \in \mathbb{R}$  such that f(x) = c.