Summer Workshop in Mathematics (SWIM)
Kannur University, May, 2025
Week 4
Tutorial problems

May 27, 2025

1. Find the eigen values and corresponding eigen vectors of the identity matrix. Do the same for an arbitrary diagonal matrix.

2. A $n \times n$ matrix A is said to be nilpotent if there exists a positive integer k such that $A^k = 0$. Prove that all the eigen values of a nilpotent matrix are zero.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator given by
$$T(v_1) = 0,\ T(v_2) = v_1,\ T(v_3) = v_1 + v_2.$$
with respect to a basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$. Prove that $T^3 = 0$.

4. Let $T : V \to V$ be a linear operator on a vector space of dimension 2. Assume that there is a non-zero vector $v \in V$ such that v is not an eigenvector for T. Prove that $\{v, T(v)\}$ is a basis of V.

5. Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by
$$T_\theta(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$
(T_θ is called the rotation by θ on the plane $\mathbb{R}^2$). Show that there is no eigen vector of T_θ corresponding to any non-zero eigen value.

6. Let A be a $n \times n$ matrix such that $A^2 = I$. Prove that only eigenvalues of A are $1, -1$.

7. Let A be a $n \times n$ matrix where sum of entries in each row is 1. Find an explicit eigenvalue of A and an eigenvector corresponding to that eigenvalue.

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8. Let A be a 2×2 real matrix with an eigen value c. Consider $A - cI$ as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that nullity of $A - cI$ is non-zero and hence $A - cI$ is not invertible.

9. Do A and A^t have the same eigenvalues? Justify your answer.

10. Two $n \times n$ matrices A and B are said to be similar if there exists an invertible matrix P such that $B = P^{-1}AP$. Prove that similar matrices have same eigenvalue.

11. If one of A and B are non-singular, prove that AB and BA have same eigenvalues.

12. Prove that a real 3×3 matrix has at least one real eigenvalue.

13. $\mathrm{SO}_3 := \{A \in M_3(\mathbb{R}) \mid AA^t = I = A^tA,\ \det(A) = 1\}$. Prove that every element $A \in \mathrm{SO}_3$ has the eigenvalue 1. (Hint: Use the relations $\det(A) = \det(A^t),\ A^t(A - I) = (I - A)^t,\ \det(-B) = -\det(B)$).

14. Let A be a 2×2 matrix, with characteristic polynomial $p(t)$. Prove that

$$p(t) = t^2 - (trA)t + detA.$$

15. Let A be a $n \times n$ skew-symmetric matrix where n is odd. Find an explicit eigenvalue of A. What is the determinant of A?

16. Let A be a $n \times n$ nilpotent matrix. Prove that $\det(A + I) = 1$.

17. Let A be a 2×2 real skew-symmetric matrix with positive off-diagonal entries. Can it be similar to a 2×2 real upper triangular matrix with one zero entry in the main diagonal?

18. Let A be a 2×2 matrix with all positive entries and $\det(A) > 0$. Prove that

$$\frac{\mathrm{trace}(A)}{2} \geq \sqrt{\det(A)}.$$

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19. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
    (a) Find the eigenvectors and eigenvalues of the matrix A.
    (b) Find a matrix P such that $P^{-1}AP$ is diagonal.
    (c) Compute A^{30}.

20. Let A be a $n \times n$ matrix. Prove that eigen vectors corresponding to distinct eigen values are linearly independent. Use this result to show the following:
if A is a 2×2 real matrix satisfying $A^2 = I$, then A is diagonalizable.

21. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Find the eigenvalues of A and corresponding eigenvectors. Is A diagonalizable?

22. Let M be a matrix made up of two diagonal blocks: $M = \begin{bmatrix} A & O \\ O & D \end{bmatrix}$. Prove that M is diagonalizable if and only if A and D are.

23. Let A be a 2×2 matrix such that $\text{trace}(A) = 0$. Further assume that off-diagonal entries of A are positive. Prove that A^2 has a single positive eigenvalue with multiplicity 2 and it commutes with every matrix, that is, $A^2 B = BA^2$ for all $B \in M_2(\mathbb{R})$.

24. Let A be a 4×4 skew-symmetric integer matrix. Prove that $\det(A) \in \mathbb{N}$.