Summer Workshop in Mathematics (SWIM)
Kannur University, May, 2025
Week 3
Tutorial problems

May 21, 2025

1. Let T be the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Prove that the image of T is determined by values of T on any fixed basis of $\mathbb{R}^n$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Prove that T is of the form

$$T(x, y) = (ax + by, cx + dy),$$

for some $a, b, c, d \in \mathbb{R}$.

3. Let T be the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Consider the map $S := T \circ T : \mathbb{R}^n \to \mathbb{R}^n$. Then show that S is a linear map.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then show that the kernel of T and image of T are subspaces of $\mathbb{R}^n$.

5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then show that T is one-one iff kernel of T=$\{0\}$ and T is onto iff the image of T=$\mathbb{R}^n$.

6. Let T be a linear isomorphism of $\mathbb{R}^n$. Then,
   (a) T^{-1} is also linear.
   (b) $T \circ T$ is also an isomorphism.

7. Let $(v_1, \cdots, v_n)$ be a subset of a vector space V over a field F. Prove that the map $T : F^n \to V$ defined by $T(x) = v_1x_1 + \cdots + v_nx_n$ is a linear transformation.

8. Prove that any finite dimensional vector space over $\mathbb{R}$ is linearly isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Using the fact that $\mathbb{Q}$ is countable conclude that $\mathbb{R}$ is not a finite dimensional vector space over $\mathbb{Q}$.

May 22, 2025

9. Is there any injective linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$?

10. Is there any surjective linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$?

11. Show that any injective linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a linear isomorphism. Is the same conclusion true if injective map is replaced by surjective map?

12. Let A be an $m \times n$ matrix. Prove that the space of solutions of the linear system $AX = 0$ has dimension at least $n - m$.

13. Let V be a n dimensional vector space and $f : V \to \mathbb{R}$ be a non-zero linear functional. Prove that $\ker(f)$ is a $n-1$ dimensional vector space. Use this result to show that the set of traceless $n \times n$ matrices has dimension $n^2 - 1$.

14. Let V be a finite dimensional vector space and $f, g : V \to \mathbb{R}$ be two non-zero linear functionals such that $\ker(f) = \ker(g)$. Show that there exists a non-zero constant c such that $g = cf$.

15. Prove that Trace is a linear functional from $M_n(\mathbb{R})$ to $\mathbb{R}$ which satisfies

$$\text{Trace}(AB - BA) = 0,$$

for all $A, B \in M_n(\mathbb{R})$. If f is a linear functional which satisfies $f(AB - BA) = 0$, for all $A, B \in M_n(\mathbb{R})$, then show that $f = c$ (Trace) for some $c \in \mathbb{R}$.

<u>May 23, 2025</u>

16. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by the rule

$$T(x, y) = (-x, y).$$

What is the matrix of T with respect to the standard bases?
(This linear map is called reflection about y-axis).

17. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by the rule

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

What is the matrix of T with respect to the standard bases?

18. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by the rule

$$T(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

What is the matrix of T with respect to the standard bases?
(T is actually called rotaion by angle θ).

19. Let P_n be the vector space given by the set of all polynomials of degree less or equal to n. Show that $\frac{d}{dx} : P_n \to P_{n-1}$ is linear and find the matrix of this linear map with respect to standard basis.

20. Find the rank of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$.

21. Find all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ which carry the line $y = x$ to the line $y = 3x$.