Summer Workshop in Mathematics (SWIM)
Kannur University, May, 2025
Week 2
Tutorial problems

May 12, 2025

1. Any subset of a linearly independent set in a vector space is linearly independent.
2. Prove that the set $\{(5,5,5)^t, (0,6,6)^t, (0,0,7)^t\}$ is linearly independent $\mathbb{R}^3$.
3. Let V be a n-dimensional vector space over a field $\mathbb{R}$. Then any linearly independent subset of V containing n elements is a basis of V.
4. Let $\{v_1, v_2\}$ be a basis of $\mathbb{R}^2$. Prove that $\{v_1+v_2, v_1-v_2\}$ is also a basis of $\mathbb{R}^2$.
5. Prove that the set $\{(1,2,0)^t, (2,1,2)^t, (3,1,1)^t\}$ is a basis of $\mathbb{R}^3$.
6. Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W.
7. Let V be a vector space of dimension n over F, and let $0 \leq r \leq n$. Prove that V contains a subspace of dimension r.

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8. Find the co-ordinate vector of $(1,0,0)$ with respect to the basis $\{(1,1,1), (0,1,1), (0,0,1)\}$ of $\mathbb{R}^3$.
9. Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is (e_2, e_1) of $\mathbb{R}^2$.
10. Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is (e_1+e_2, e_1-e_2) of $\mathbb{R}^2$.
11. Fix $\theta \in \mathbb{R}$. Prove that $\{(\cos\theta, \sin\theta).(-\sin\theta, \cos\theta)\}$ is a basis of $\mathbb{R}^2$. Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is $\{(\cos\theta, \sin\theta).(-\sin\theta, \cos\theta)\}$ of $\mathbb{R}^2$. Is this matrix orthogonal? Is the set of all real orthogonal 2×2 matrices a vector space over $\mathbb{R}$?
12. Previously we have seen that $\beta_1 := \{(1,1,1), (0,1,1), (0,0,1)\}$ and $\beta_2 := \{(1,2,0), (2,1,2), (3,1,1)\}$ are bases of $\mathbb{R}^3$. Determine the matrix P of change of basis when the old basis is β_1 and the new basis is β_2.

13. Let A be an $n \times n$ matrix with entries in a field F. Denote the i-th column of A by v_i. For any column vector $X = (x_1, \cdots, x_n)^t$, the matrix product $AX = v_1x_1 + \cdots + v_nx_n$ is a linear combination of the set $(v_1, \cdots, v_n)$. Prove the following:

    (a) The set $\{v_1, v_2, \cdots, v_n\}$ is linearly independent if and only if the only solution of the equation $AX = 0$ is the trivial solution $X = 0$.

    (b) The columns of A form a basis of F^n if A is invertible.

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14. Let W_1, W_2 be two subspaces of a 6 dimensional vector space with $\dim(W_1) = 2$ and $\dim(W_2) = 3$. What are possible dimensions of $W_1 + W_2$?

15. The subspaces $W_1, W_2, \cdots, W_n$ of a vector space V are called independent if

$$w_1 + w_2 + \cdots + w_n = 0, w_i \in W_i \text{ implies } w_i = 0, \text{ for all } i.$$

    Prove that two subspaces W_1, W_2 are independent if and only if $W_1 \cap W_2 = \{0\}$.

16. Let W, W' be two vector subspaces of a finite dimensional vector space V. Prove that $\dim(W \oplus W') = \dim(W) + \dim(W')$.

17. Let $v_1, v_2, \cdots, v_n$ be non-zero vectors in a vector space V and let W_i be the span of the single vector v_i.The subspaces $W_1, W_2, \cdots, W_n$ are independent if and only if $\{v_1, v_2, \cdots, v_n\}$ are linearly independent vectors.

18. $\mathcal{F} :=$ The set all functions from $\mathbb{R}$ to $\mathbb{R}$.
    $\mathcal{E} :=$ The set all even functions from $\mathbb{R}$ to $\mathbb{R}$.
    $\mathcal{O} :=$ The set all odd functions from $\mathbb{R}$ to $\mathbb{R}$.
    Prove that $\mathcal{F}$, $\mathcal{E}$,$\mathcal{O}$ are vector spaces over $\mathbb{R}$ and $\mathcal{F} = \mathcal{E} \oplus \mathcal{O}$.

19. Prove that the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices is the direct sum of the spaces of symmetric matrices ($A = A^t$) and of skew-symmetric matrices ($A = -A^t$).

20. Let W be the space of $n \times n$ matrices whose trace is zero. Find a subspace W' so that $M_n(\mathbb{R}) = W \oplus W'$.

21. Let UT=the set of all upper traingular 2×2 matrices and LT=the set of all 2×2 lower traingular matrices. Prove that both UT and LT are vector subspaces of $M_2(\mathbb{R})$. Find their dimensions. Is it true that $M_2(\mathbb{R}) = \text{UT} \oplus \text{LT}$?