Summer Workshop in Mathematics (SWIM) Kannur University, May, 2025 Week 2 Tutorial problems

May 12, 2025

- 1. Any subset of a linearly independent set in a vector space is linearly independent.
- 2. Prove that the set $\{(5,5,5)^t, (0,6,6)^t, (0,0,7)^t\}$ is linearly independent \mathbb{R}^3 .
- 3. Let V be a n-dimensional vector space over a field \mathbb{R} . Then any linearly independent subset of V containing n elements is a basis of V.
- 4. Let $\{v_1, v_2\}$ be a basis of \mathbb{R}^2 . Prove that $\{v_1 + v_2, v_1 v_2\}$ is also a basis of \mathbb{R}^2 .
- 5. Prove that the set $\{(1,2,0)^t, (2,1,2)^t, (3,1,1)^t\}$ is a basis of \mathbb{R}^3 .
- 6. Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations AX = 0, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W.
- 7. Let V be a vector space of dimension n over F, and let $0 \le r \le n$. Prove that V contains a subspace of dimension r.

May 13, 2025

- 8. Find the co-ordinate vector of (1, 0, 0) with respect to the basis $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ of \mathbb{R}^3 .
- 9. Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is (e_2, e_1) of \mathbb{R}^2 .
- 10. Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is $(e_1 + e_2, e_1 e_2)$ of \mathbb{R}^2 .
- 11. Fix $\theta \in \mathbb{R}$. Prove that $\{(\cos \theta, \sin \theta).(-\sin \theta, \cos \theta)\}$ is a basis of \mathbb{R}^2 . Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is $\{(\cos \theta, \sin \theta).(-\sin \theta, \cos \theta)\}$ of \mathbb{R}^2 . Is this matrix orthogonal? Is the set of all real orthogonal 2×2 matrices a vector space over \mathbb{R} ?
- 12. Previously we have seen that $\beta_1 := \{(1,1,1), (0,1,1), (0,0,1)\}$ and $\beta_2 := \{(1,2,0), (2,1,2), (3,1,1)\}$ are bases of \mathbb{R}^3 . Determine the matrix P of change of basis when the old basis is β_1 and the new basis is β_2 .

- 13. Let A be an $n \times n$ matrix with entries in a field F. Denote the *i*-th column of A by v_i . For any column vector $X = (x_1, \dots, x_n)^t$, the matrix product $AX = v_1x_1 + \dots + v_nx_n$ is a linear combination of the set (v_1, \dots, v_n) . Prove the following:
 - (a) The set $\{v_1, v_2, \dots, v_n\}$ is linearly independent if and only if the only solution of the equation AX = 0 is the trivial solution X = 0.
 - (b) The columns of A form a basis of F^n if A is invertible.

May 16, 2025

- 14. Prove or verify the following exercises regarding direct sum of two vector spaces.
 - (a) Let W, W' be two vector subspaces of a finite dimensional vector space V. Prove that $\dim(W \oplus W') = \dim(W) + \dim(W')$.
 - (b) Prove that the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices is the direct sum of the spaces of symmetric matrices $(A = A^t)$ and of skew-symmetric matrices $(A = -A^t)$.
 - (c) Let W be the space of $n \times n$ matrices whose trace is zero. Find a subspace W' so that $M_n(\mathbb{R}) = W \oplus W'$.
 - (d) Let UT=the set of all upper traingular 2×2 matrices and LT=the set of all 2×2 lower traingular matrices. Prove that both UT and LT are vector subspaces of $M_2(\mathbb{R})$. Find their dimensions. Is it true that $M_2(\mathbb{R}) = \text{UT} \oplus \text{LT}$?