Summer Workshop in Mathematics (SWIM)
Kannur University, May, 2025
Week 2
Tutorial problems

May 12-13, 2025

1. Any subset of a linearly independent set in a vector space is linearly independent.
2. Prove that the set $\{(5,5,5)^t, (0,6,6)^t, (0,0,7)^t\}$ is linearly independent $\mathbb{R}^3$.
3. Let V be a n-dimensional vector space over a field $\mathbb{R}$. Then any linearly independent subset of V containing n elements is a basis of V.
4. Let $\{v_1, v_2\}$ be a basis of $\mathbb{R}^2$. Prove that $\{v_1 + v_2, v_1 - v_2\}$ is also a basis of $\mathbb{R}^2$.
5. Prove that the set $\{(1,2,0)^t, (2,1,2)^t, (3,1,1)^t\}$ is a basis of $\mathbb{R}^3$.
6. Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W.
7. Let V be a vector space of dimension n over F, and let $0 \leq r \leq n$. Prove that V contains a subspace of dimension r.
8. Let A be an $n \times n$ matrix with entries in a field F. Denote the i-th column of A by v_i. For any column vector $X = (x_1, \cdots, x_n)^t$, the matrix product $AX = v_1x_1 + \cdots + v_nx_n$ is a linear combination of the set $(v_1, \cdots, v_n)$. Prove the following:
   (a) The set $\{v_1, v_2, \cdots, v_n\}$ is linearly independent if and only if the only solution of the equation $AX = 0$ is the trivial solution $X = 0$.
   (b) The columns of A form a basis of F^n if A is invertible.
9. Is the set of all real orthogonal 2×2 matrices a vector space over $\mathbb{R}$?

May 16, 2025

10. Prove or verify the following exercises regarding direct sum of two vector spaces.
   (a) Let W, W' be two vector subspaces of a finite dimensional vector space V. Prove that $\dim(W \oplus W') = \dim(W) + \dim(W')$.
   (b) Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is $(e_1 + e_2, e_1 - e_2)$ of $\mathbb{R}^2$.

(c) Prove that the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices is the direct sum of the spaces of symmetric matrices ($A = A^t$) and of skew-symmetric matrices ($A = -A^t$).

(d) Let W be the space of $n \times n$ matrices whose trace is zero. Find a subspace W' so that $M_n(\mathbb{R}) = W \oplus W'$.

(e) Let UT=the set of all upper traingular 2×2 matrices and LT=the set of all 2×2 lower traingular matrices. Prove that both UT and LT are vector subspaces of $M_2(\mathbb{R})$. Find their dimensions. Is it true that $M_2(\mathbb{R}) = \mathrm{UT} \oplus \mathrm{LT}$?