SWIM@KSoM
May 5-30, 2025 at Kannur University (Thavakkara Campus)

Analysis Exercises

A Derivative: definition, properties.

1. Show that $f(x) = \begin{cases} x^2+x+1, & x \le 1 \\ 3x, & x > 1 \end{cases}$ is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f .

2. (a) Does the graph of $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ have a tangent at the origin? Give reasons for your answer.

   (b) Does the graph of $g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ have a tangent at the origin? Give reasons for your answer.

3. Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2; & \text{if } x \text{ is rational} \\ 2x-1, & \text{if } x \text{ is irrational} \end{cases}$$

   Is f differentiable at 1. If so find $f'(1)$.

4. Find all values of x at which the tangent line to the given curve satisfies the stated property.

   (a) $y = \dfrac{x^2+1}{x+1}$; parallel to the line $y = x$

   (b) $y = \dfrac{1}{x+4}$; passes through the origin

   (c) $y = \dfrac{2x+5}{x+2}$; y-intercept 2.

   (d) $y = \dfrac{\sqrt{12-3x^2}}{2}$; perpendicular to $y = 0$.

5. Determine if the piecewise defined function $g(x) = \begin{cases} x^{2/3}, & x \ge 0 \\ x^{1/3}, & x < 0 \end{cases}$ is differentiable at the origin.

6. (a) Graph the function $f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$

   (b) Is f continuous at $x = 0$?

   (c) Is f differentiable at $x = 0$?

7. (a) Graph the function $f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$

   (b) Is f continuous at $x = 0$?

   (c) Is f differentiable at $x = 0$?

8. Suppose that $f(x) = \begin{cases} x^2-1, & x \le 1 \\ k(x-1), & x > 1. \end{cases}$ For what values of k is f continuous? For what values of k is f differentiable?

9. Find the values of a and b that make the following function differentiable for all x-values. $f(x) = \begin{cases} ax+b, & x > -1 \\ bx^2-3, & x \le -1 \end{cases}$

10. A function f is defined as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \le c, \\ ax+b & \text{if } x > c, \end{cases} \quad (a,\ b,\ c \text{ constants}).$$

    Find values of a and b (in terms of c) such that $f'(c)$ exists.

11. Solve the above question when $f(x) = \begin{cases} \dfrac{1}{|x|} & \text{if } |x| > c, \\ a + bx^2 & \text{if } |x| \le c. \end{cases}$

12. For what value or values of the constant m, if any, is $f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0. \end{cases}$

(a) continuous at $x = 0$?

(b) differentiable at $x = 0$?

13. Suppose that a function f is differentiable at $x = 1$ and $\lim_{h\to 0} \frac{f(1+h)}{h} = 5$. Find $f(1)$ and $f'(1)$.

14. Suppose that a function f is differentiable at $x = 2$ and $\lim_{x\to 2} \frac{x^3 f(x)-24}{x-2} = 28$. Find $f(2)$ and $f'(2)$.

15. Show that the following functions are differentiable at $x = 0$.

(a) $|x| \sin x$

(b) $x^{2/3} \sin x$

(c) $\sqrt[3]{x}(1 - \cos x)$

(d) $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

16.* Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f^{'}(0)$.

17. Is the derivative of $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$?

18.* Recall that a function f is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$, for all x in the domain of f . Assuming that f is differentiable, prove:

(a) f' is odd if f is even

(b) f' is even if f is odd.

19.* Given that the derivative $f'(a)$ exists. State which of the following statements are true.

(a) $f'(a) = \lim_{h\to a} \dfrac{f(h) - f(a)}{h - a}$.

(b) $f'(a) = \lim_{h\to 0} \dfrac{f(a) - f(a - h)}{h}$.

(c) $f'(a) = \lim_{t\to 0} \dfrac{f(a + 2t) - f(a)}{t}$.

(d) $f'(a) = \lim_{t\to 0} \dfrac{f(a + 2t) - f(a + t)}{2t}$.

20. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that $g(x) := |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

21. (a) If

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x^2 + 1, & x > 0 \end{cases}$$

Show that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$$

but that $f'(0)$ does not exist.

(b) If

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$$

Show that $f'(0)$ exists but $f''(0)$ does not exist.

B Basic differentiation rules (sum, product, quotient, chain rule)

22. (a) Show that if f and g are functions for which $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x, then $f^2(x) + g^2(x)$ is a constant.

(b) Show that if f and g are functions for which $f'(x) = g(x)$ and $g'(x) = f(x)$ for all x, then $f^2(x) - g^2(x)$ is a constant.

(c) Give an example of functions f and g with this property.

23. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	-2	5	7
5	3	-1	12	4

(a) $F'(3)$, where $F(x) = f(g(x))$

(b) $G'(3)$, where $G(x) = g(f(x))$

24. (a) Find $f'(x^2)$ if $\dfrac{d}{dx}\left[f(x^2)\right] = x^2$.

(b) Find $\dfrac{d}{dx}[f(x)]$, if $\dfrac{d}{dx}[f(3x)] = 6x$.

C Local maxima and minima, Fermat's Theorem.

25.* Find the relative extrema of the function f.

(a) $f(x) = \dfrac{x+3}{x-2}$

(b) $f(x) = |3x - x^2|$

26. Find the absolute maximum and minimum values of f , if any, on the given interval, and state where those values occur.

(a) $f(x) = x^2 - x - 2$; $(-\infty, +\infty)$

(b) $f(x) = x^4 + 4x$; $(-\infty, +\infty)$

(c) $f(x) = x^3 - 9x + 1$; $(-\infty, +\infty)$

(d) $f(x) = \dfrac{x^2+1}{x+1}$; $(-5, -1)$

(e) $f(x) = \dfrac{x-2}{x+1}$; $(-1, 5]$

27. Find the absolute maximum and minimum values of attained on $\left[\frac{1}{2}, \frac{7}{2}\right]$ by the function

$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x-2)(x-3), & x \geq 1. \end{cases}$$

28. (a) Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

(b) Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point (1,2).

(c) Determine the values of constants a, b, c and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point (0,0) and a local minimum at the point (1,-1).

29. (a) A piece of wire 40 cm long is cut into two pieces. One piece is bent into the shape of a square and the other is bent into the shape of a circle. How should the wire be cut so that the total area enclosed is a (a) maximum and (b) minimum?

(b) Express 18 as a sum of two positive numbers such that the product of the first and square of the second is as large as possible.

30. Find the absolute maximum value of $f(x) = x^2 \ln\left(\dfrac{1}{x}\right)$ and say where it is assumed.

D Rolle's Theorem, Mean Value Theorem

31. Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

(a) $f(x) = x^2 - 8x + 15$; $[3, 5]$

(b) $f(x) = \cos x$; $[\pi/2, 3\pi/2]$

(c) $f(x) = \dfrac{(x^2-1)}{(x-2)}$; $[-1, 1]$

(d) $f(x) = 2^x - 30x + 82$; $[3, 7]$

32. Show that if f is differentiable on an open interval and $f'(x) \neq 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

33. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.

34. Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

(a) $f(x) = x^2 - x;\quad [-3, 5]$

(b) $f(x) = \sqrt{25 - x^2};\quad [-5, 3]$

(c) $f(x) = x - \dfrac{1}{x};\quad [3, 4]$

(d) $f(x) = -2x^3 + 6x - 2;\quad [-3, 5]$

35. Let $f(x) = x^{\frac{2}{3}}$, $a = -1$, and $b = 8$.
   (a) Show that there is no point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
   (b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

36. (a) Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1,\ t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.
   (b) What is the geometric significance of the result in part (a).

37. An automobile travels 4 km along a straight road in 5 min. Show that the speedometer reads exactly 48 km/h at least once during the trip.

38. Use the Mean-Value Theorem to show that $\sqrt{y} - \sqrt{x} < \dfrac{y - x}{2\sqrt{x}}$ if $0 < x < y$.

39.* Prove the following result using the Mean-Value Theorem: Let f be continuous at x_0 and suppose that $\lim_{x \to x_0} f'(x)$ exists. Then f is differentiable at x_0 and $f'(x_0) = \lim_{x \to x_0} f'(x)$ [Hint: The derivative $f'(x_0)$ is given by $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ provided this limit exists.]

40. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?

41. (a) Suppose that $0 < f'(x) < \dfrac{1}{2}$ for all x-values. Show that $f(-1) < f(1) < f(-1) + 1$.
   (b) Show that $|\cos x - 1| \leq |x|$ for all x-values.(Hint: Consider $f(t) = \cos t$ on $[0, x]$.)
   (c) Show that for any numbers a and b, the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.

42. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that

$$f'(c) = \lim_{n \to \infty} (n\{f(c + 1/n) - f(c)\}).$$

   However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

43. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{N}$. Fix $a \in \mathbb{R}$. Find

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a}.$$

44. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Let $a < x_n < c < y_n < b$ be such that $y_n - x_n \to 0$. Show that

$$\lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c)$$

45. Use the Mean Value Theorem to prove that
   (a) $|\sin x - \sin y| \leq |x - y|$ for all x, y in $\mathbb{R}$.
   (b) $(x - 1)/x < \ln x < x - 1$ for $x > 1$.

46. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x - y)^2$ for all x, y. Show that f is differentiable, and the derivative is zero. Hence conclude that f is a constant.

47. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is bounded. Prove that f is uniformly continuous.

48.* Let $f : (0, 1] \to \mathbb{R}$ be differentiable with $|f'(x)| < 1$. Define $a_n := f(1/n)$. Show that (a_n) converges.

49. Let $f : [a, b] \to [a, b]$ be differentiable. Assume that $f'(x) \neq 1$ for $x \in [a, b]$. Prove that f has a unique fixed point in $[a, b]$.

50.* Let $f \colon [a, b] \to \mathbb{R}$ be a differentiable function. Assume that there is no $x \in [a, b]$ such that $f(x) = f'(x) = 0$. Prove that the set $\{t \in [a, b] \colon f(t) = 0\}$ is finite.

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