KSOM-SWIM 2025: ANALYSIS:

1. THE $\varepsilon - \delta$ DEFINITION OF CONTINUITY

Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. We say that f is continuous at a point $x_0 \in X$ if given any $\varepsilon > 0$, there is a $\delta > 0$ such that for any $x \in [x_0 - \delta, x_0 + \delta] \cap X$ we have $f(x) \in [f(x_0) - \varepsilon, f(x_0) + \varepsilon]$. If f is continuous at all $x_0 \in X$, we say that f is continuous on X.

▶ Observe that δ will depend on ε and may also depend on x_0. For a fixed x_0, the smaller the ε, smaller will have to be δ. This has the implication that it suffices to check the condition for $0 < \varepsilon \le 1$ or $0 < \varepsilon \le 0.1$ or $0 < \varepsilon \le 0.0001$. Why? Whatever δ works for $\varepsilon = 0.1$ works for any $\varepsilon > 0.1$ too.

▶ How to show that f is not continuous at x_0? We must find some $\varepsilon > 0$ such that for any $\delta > 0$, there is some $x \in [x_0 - \delta, x_0 + \delta] \cap X$ such that $f(x) \notin [f(x_0) - \varepsilon, f(x_0) + \varepsilon]$. Here also, it suffices to do this for $\delta \le 1$ or $\delta \le 0.1$ etc., because if $\delta > 0.1$, then any point in $[x_0 - 0.1, x_0 + 0.1]$ is also in $[x_0 - \delta, x_0 + \delta]$. So we can take the same point for larger δ.

Problem 1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Show that f is continuous.

Solution Fix $x_0 \in \mathbb{R}$. Fix any $0 < \varepsilon \le 1$. Choose $\delta = \frac{\varepsilon}{2|x_0|+1}$. Clearly $0 < \delta \le \varepsilon \le 1$. Then if $x \in [x_0 - \delta, x_0 + \delta]$, we can write

$$|x^2 - x_0^2| = |x - x_0| \times |x + x_0| \le \delta|x + x_0| \le \delta(|x| + |x_0|).$$

Since $x \in [x_0 - \delta, x_0 + \delta]$ and $|x_0 - \delta| \le |x_0| + \delta$ and $|x_0 + \delta| \le |x_0| + \delta$, we also get $|x| \le |x_0| + \delta$. Therefore,

$$|x^2 - x_0^2| \le \delta(2|x_0| + \delta) \le \delta(2|x_0| + 1) \le \varepsilon.$$

This completes the proof for $\varepsilon \le 1$. What if $\varepsilon > 1$? Then we can take the same δ that worked for $\varepsilon = 1$, namely $\delta = \frac{1}{2|x_0|+1}$. Then if $|x - x_0| \le \delta$, we have $|f(x) - f(x_0)| \le 1 \le \varepsilon$.

Problem 2. Define $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 7 & \text{if } x = 0. \end{cases}$. Show that f is not continuous at 0.

Solution Let $\varepsilon = 1$. Take any $\delta > 0$. Let $x = \frac{\delta}{2}$ which is clearly in $[0 - \delta, 0 + \delta]$. But $f(x) = \frac{1}{x} = \frac{2}{\delta}$. Now if $\delta \le 1/5$, then $f(x) \ge 10$ and hence $|f(x) - f(0)| > 10 - 7 > 1$.

What if $\delta > \frac{1}{5}$? We can take the same point that worked for $\delta = 1/5$, namely $x = 1/10$. Then $|x| = \frac{1}{10} < \delta$ and $|f(x) - f(0)| = 3 > 1$.

Thus for any δ we have found a point $x \in [-\delta, \delta]$ such that $|f(x) - f(0)| > 1$. Hence f is not continuous at 0.

Problem 3. Let $f(x) = x^3$ for $x \in \mathbb{R}$. Show that f is continuous everywhere.

Problem 4. Let $f(x) = \lfloor x \rfloor$. Show that f is continuous at all $x \in \mathbb{R} \setminus \mathbb{Z}$ and that f is not continuous at any $x \in \mathbb{Z}$. What about $g(x) = \frac{1}{5}\lfloor x \rfloor$?

Problem 5. Define $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$. Show that f is not continuous at any point of $\mathbb{R}$.

Problem 6. Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$. What are the continuity points of f?

2. Sequential definition of continuity

Checking the continuity of a function using the $\varepsilon - \delta$ definition is somewhat painful. Much simpler is the following sequential criterion.

Proposition 7. *Let $f : X \to \mathbb{R}$ and let $x_0 \in X$. Then f is continuous at x_0 (according to the $\varepsilon - \delta$ definition) if and only if $f(x_n) \to f(x_0)$ for every sequence (x_n) in X such that $x_0 \to x$.*

If we are already familiar with sequences and their limits, then it is often easier to check the continuity or discontinuity of a function at a point using this sequential criterion.

Problem 8. Define $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$. Show that f is not continuous at any point of $\mathbb{R}$.

Solution Fix $x_0 \in \mathbf{Q}$. As irrational numbers are dense in $\mathbb{R}$, there exist $x_n \in \mathbb{Q}^c$ such that $x_n \to x_0$. But $f(x_n) = 0$ and $f(x_0) = 1$, therefore $f(x_n) \not\to f(x_0)$. This shows that f is not continuous at x_0. Hence f is not continuous at any rational point.

Next fix $x_0 \in \mathbb{Q}^c$. As rational numbers are dense in $\mathbb{R}$, there exist $x_n \in \mathbb{Q}$ such that $x_n \to x_0$. But $f(x_n) = 1$ and $f(x_0) = 0$, therefore $f(x_n) \not\to f(x_0)$. This shows that f is not continuous at x_0. Hence f is not continuous at any irrational point.

Putting these together, f is not continuous at any point of $\mathbb{R}$.

Problem 9. Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$. Show that 0 is the only continuity point of f.

Problem 10. Let $f(x) = \begin{cases} x(x-1)(x+3) & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$. Find all the continuity points of f.

Problem 11. Fix a natural number n and let $f(x) = x^n$. Show that f is continuous.

Problem 12. Given $a_1, \dots, a_n \in \mathbb{R}$, construct a function $f : \mathbb{R} \to \mathbb{R}$ whose only discontinuity points are $a_1, \dots, a_n$.

Proof of Proposition 7. Suppose f is continuous at x_0. Let (x_n) be a sequence in X such that $x_n \to x_0$. We must show that $f(x_n) \to f(x_0)$. To do this, fix $\varepsilon > 0$ and find a $\delta > 0$ such that $|f(x) - f(x_0)| \le \varepsilon$ if $|x - x_0| \le \delta$. Then find N such that $|x_n - x_0| \le \delta$ if $n \ge N$. Putting these two together, we see that $|f(x_n) - f(x_0)| \le \varepsilon$ if $n \ge N$. As this works for any $\varepsilon > 0$, we see that $f(x_n) \to f(x_0)$.

Suppose f is not continuous at x_0. Then there is some $\varepsilon > 0$ such that for any $\delta > 0$, there is some $x \in [x_0 - \delta, x_0 + \delta]$ such that $|f(x) - f(x_0)| > \varepsilon$. In particular, for $\delta = \frac{1}{n}$, there is some $x_n \in [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}|$ such that $|f(x_n) - f(x_0)| > \varepsilon$. From this it is clear that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$. This completes the proof of the reverse implication[1]. ■

3. Algebraic properties of continuous functions

Although the sequential criterion is more convenient to check continuity of a function, most often we do not even have to go through that difficulty. We already know many functions (powers, exponential, logarithm, trigonometric, etc.) that are continuous on their domains. If we combine them in simple ways by adding, multiplying, dividing or composing, the resulting functions are continuous. Thus to check continuity, we can simply invoke these properties.

Proposition 13. *Let $f, g : X \to \mathbb{R}$. Suppose both f and g are continuous at $x_0 \in \mathbb{R}$. Then the following functions are also continuous at x_0: (a) Linear combination $af + bg$ where $a, b \in \mathbb{R}$, (b) Product $f.g$ (c) Quotient f/g* provided $g(x_0) \neq 0$ *(d) Maximum $f \vee g$, (e) Minimum $f \wedge g$.*

Note that the quotient function f/g is only defined on the subdomain $X' = \{x : g(x) \neq 0\}$. Unless $x_0 \in X'$, we cannot even talk about continuity of f/g at x_0.

Another useful operation is composition.

Proposition 14. *Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Assume that Range$(f) \subseteq Y$ (in other words, $f : X \to Y$). Let $x_0 \in X$. Suppose f is continuous at x_0 and g is continuous at $f(x_0) \in Y$, then $g \circ f : X \to \mathbb{R}$ is continuous at X.*

Problem 15. All polynomials with real coefficients are continuous on $\mathbb{R}$.

Solution Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ where $a_k \in \mathbb{R}$. Let us use induction on n. We use two known facts: The constant function $f(x) = 1$ and the identity function $g(x) = x$ are continuous.

Base case $n = 0$: Then $p(x) = a_0$. We can write $p(x) = a_0 f$, hence p is continuous.

Assume that polynomials of degree less than n are continuous. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ be a polynomial of degree n. Then we can write $p(x) = xq(x) + a_0$ where $q(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \dots + a_1$. In other words, $p(x) = g(x)q(x) + a_0 f(x)$. We already know that f, g are continuous and by induction hypothesis q is continuous. Therefore, p is also continuous.

[1]A implies B is equivalent to Not B implies Not A. What we showed is that if the $\varepsilon - \delta$ condition fails, then the sequential condition fails. Therefore, the sequential condition implies the $\varepsilon - \delta$ condition.

Problem 16. Show that the function 2^{x^2-3x} is continuous on $\mathbb{R}$.

SolutionLet $f(x) = x^2-3x$ and $g(x) = 2^x$. We know that f and g are continuous (f is a polynomial and we showed above; for g read the book). The given function is just $g \circ f$, hence also continuous.

Problem 17. Show that the following functions are continuous on $\mathbb{R}$: $\sin(x^2+x-2)$, $\sqrt{(x+3)^2+17}$, $\frac{x^2-3x+2}{x^2+1}$.

Problem 18. Show that the following functions are continuous on $\mathbb{R}$ on their domain of definition (which you should specify explicitly): $\frac{1}{x^2}$, $\sin(1/x)$, $\frac{x^2-3x+2}{x^2-1}$.

Proof of Proposition 13. All the implications follow from the sequential criterion for continuity and the corresponding statements for limits of sequences, namely: If $u_n \to u$ and $v_n \to v$, then (a) $au_n + bv_n \to au + bv$ for any $a, b \in \mathbb{R}$, (b) $u_n v_n \to uv$, (c) $\frac{u_n}{v_n} \to \frac{u}{v}$ provided $v \neq 0$, (d) $u_n \vee v_n \to u \vee v$, (e) $u_n \wedge v_n \to u \wedge v$.

For example, if $x_n \to x_0$, then we know that $f(x_n) \to f(x_0)$ (because f is continuous at x_0) and $g(x_n) \to g(x_0)$ (because g is continuous at x_0). Applying (b) above, we see that $f(x_n)g(x_n) \to f(x_0)g(x_0)$. Thus fg is continuous at x_0. We leave the rest as exercises. ■

4. Continuous functions on closed bounded intervals

A closed and bounded interval in one of the form $X = [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. Note that $[0, \infty)$ is not bounded and and $(0, 1)$ is not closed. Continuous functions on closed, bounded intervals have special properties not shared by other domains.

Theorem 19. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function.*

(1) Let $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$. Then the range of f is equal to $[m, M]$.

(2) f is uniformly continuous. That is, given any $\varepsilon > 0$, there is a $\delta > 0$ (depending only on ε) such that for any $x, y \in [a, b]$ with $|x - y| \le \delta$, we have $|f(x) - f(y)| \le \varepsilon$.

The first statement says multiple things. Let us break it down.

(1) f attains its maximum and minimum. That is, there exists $c, d \in [a, b]$ such that $f(c) = M$ and $f(d) = m$.

(2) Intermediate value theorem: For every $v \in [\min f, \max f]$, there is a point $x \in [a, b]$ such that $f(x) = v$. Often it is stated only for v lying between $f(a)$ and $f(b)$, but it is equivalent to what we have stated.

Uniform continuity says that for any $\varepsilon > 0$, the δ required in the $\varepsilon - \delta$ definition of continuity can be chosen to not depend on the point x_0.

Example 20. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. When we showed continuity, we said that for fixed x_0 and $\varepsilon > 0$, if we take $\delta = \frac{\varepsilon}{2|x_0|+1}$, then $|f(x) - f(x_0)| \leq \varepsilon$ for $|x - x_0| \leq \delta$. Here the δ chosen appears to depend on x_0.

But if we restrict f to an interval like $[0, 10]$, then we can take $\delta = \frac{\varepsilon}{25}$. Then for any $x, y \in [0, 10]$,

$$|f(y) - f(x)| = |y - x||y + x| \leq 20\delta < \varepsilon.$$

Thus, we have one δ which works for all $x \in [0, 10]$. So f is uniformly continuous on $[0, 10]$.

But f is not uniformly continuous on $\mathbb{R}$. Then for any $x > 0$ and $\delta > 0$ we have

$$f(x + \delta) - f(x) = \delta(2x + \delta) > 2x\delta.$$

Therefore if $x > \frac{\varepsilon}{2\delta}$, then $f(x + \delta) - f(x) > \varepsilon$. Thus, for any $\delta > 0$, there are points x for which $|f(x + \delta) - f(x)|$ is not smaller than ε. So f is not uniformly continuous.

Sequential criterion for uniform continuity Let $f : X \to \mathbb{R}$. Then f is uniformly continuous on X if and only if for any sequences $(x_n), (y_n)$ in X with $x_n - y_n \to 0$, we have $f(x_n) - f(y_n) \to 0$.

Example 21. If $f : (0, 1) \to \mathbb{R}$ is defined as $f(x) = \frac{1}{x}$, then we claim that f is not uniformly continuous. Let us use the sequential criterion. Let $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Then $|y_n - x_n| = \frac{1}{n(n+1)} \to 0$ but $f(y_n) - f(x_n) = 1$ which does not go to 0. Hence f is not uniformly continuous.

Proof that f attains its minimum and maximum. By definition of infimum, for any $n \geq 1$, there is some $x_n \in [a, b]$ such that $m \leq f(x_n) \leq m + \frac{1}{n}$. By the Bolzano-Weierstrass property, (x_n) has a convergent subsequence (x_{n_k}) that converges to some point $u \in [a, b]$. By continuity of f, we see that $f(u) = \lim f(x_n) = m$. Thus f attains its minimum.

Adapt the same ideas to show that f attains its maximum. ■

Proof of the intermediate value theorem. Assume that $f(a) \leq f(b)$ and suppose $v \in [f(a), f(b)]$. Define $c = \sup\{x \in [a, b] : f(x) \leq v\}$. As $f(a) \leq v$, the set is non-empty and bounded above by b, hence $c \in [a, b]$. By definition of supremum, $c + \frac{1}{n}$ does not belong to the set. Hence $f(c + \frac{1}{n}) > v$. Let $n \to \infty$ and use continuity of f to see that $f(c) \geq v$. Further, by definition of supremum, for any n there is some $x_n \in [c - \frac{1}{n}, c] \cap \{x \in [a, b] : f(x) \leq v\}$. This is a complicated way of saying that $x_n \in [c - \frac{1}{n}, c]$ and $f(x_n) \leq v$. Therefore $x_n \to c$ and hence $f(x_n) \to f(c)$. Therefore, $f(c) \leq v$. As $f(c) \leq v$ and $f(c) \geq v$, we see that $f(c) = v$.

What is $f(a) > f(b)$? Define $c = \inf\{x \in [a, b] : f(x) \leq v\}$ and carry out the rest of the argument as above to show that $f(c) = v$.

We have shown that if $v \in [f(a), f(b)]$, then there is some c such that $f(c) = v$. But the theorem allows $v \in [m, M]$ (a larger interval than $[f(a), f(b)]$). But the previous point already tells us that the min and max are attained, so there exist $a', b' \in [a, b]$ such that $f(a') = m$ and $f(b') = M$. By what we have shown, on the interval $[a', b']$ (or $[b', a']$ if $b' \leq a'$) f attains all values between $f(a') = m$ and $f(b') = M$. But $[a', b'] \subseteq [a, b]$, therefore, on the interval $[a, b]$ also f attains all values in $[m, M]$. ■

Proof of uniform continuity. If f is not uniformly continuous, then there is some $\varepsilon > 0$ such that for any n there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| \leq \frac{1}{n}$ but $|f(x_n) - f(y_n)| > \varepsilon$. Then $|x_n - y_n| \to 0$

but $|f(x_n) - f(y_n)| \not\to 0$. By Bolzano-Weierstrass' theorem, we can find a subsequence (n_k) such that $x_{n_k} \to x$ and $y_{n_k} \to y$ and $f(x_{n_k}) \to u$ and $f(y_{n_k}) \to v$ for some $x, y \in [a, b]$ and some $u, v \in \mathbb{R}$. But $|x_{n_k} - y_{n_k}| \to 0$, hence $x = y$. Also $f(x_{n_k}) - f(y_{n_k}) \not\to 0$ and hence $u \neq v$. But by continuity of f, we have $u = f(x)$ and $v = f(y)$. Since $x = y$ we must have $u = v$, contradicting that $u \neq v$. ■

Problem 22. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then show that f has a root in $[a, b]$.

Problem 23. Show that any third degree polynomial with real coefficients has a real root. For what other degrees can we reach the same conclusion?

Problem 24. Let $f(x) = \sin(1/x)$ for $x \in (0, 1)$. Show that f is not uniformly continuous.

Problem 25. Let $f(x) = \frac{1}{x}$ for $x > 0$. Show that f is uniformly continuous on $[1, \infty)$.

Problem 26. Find a bounded function on $\mathbb{R}$ that is not uniformly continuous.

5. Some consequences of intermediate value theorem

▶ Square root function exists: For every $x \geq 0$, there is a unique non-negative number y such that $y^2 = x$. This y is usually denoted $\sqrt{x}$ or $x^{\frac{1}{2}}$.

Proof. Fix $x_0 \geq 0$ and consider the interval $[0, M]$ where $M^2 > x$. Define $f : [0, M] \to \mathbb{R}$ by $f(x) = x^2$. Then $f(0) = 0$ and $f(M) = M^2 > x_0$. As f is continuous, by the intermediate value theorem, there must be a y such that $f(y) = x_0$. That is $y^2 = x_0$. This shows the existence of a square root. The uniqueness comes from the strict increasing property of f: If $y_1 < y_2$, then $f(y_1) < f(y_2)$, hence there cannot be two pre-images for x_0. ■

▶ For any $n \geq 1$, there is a unique nth root. The proof is similar and left as exercise.

▶ Let B be a "nice" bounded set in $\mathbb{R}^2$. Then there is a vertical line $x = x_0$ that partitions B into two parts having equal area.

Proof. Assume that $B \subseteq [-M, M]^2$ for some M (possible because B is bounded). Define $f : [-M, M] \to \mathbb{R}$ by $f(z) = \text{area}(B \cap \{(x, y) : x \leq z\})$. Then $f(-M) = 0$ and $f(M) = \text{area}(B)$. Since $y = \frac{1}{2}\text{area}(B)$ lies between $f(-M)$ and $f(M)$, intermediate value theorem would tell us that there is some z such that $f(z) = \frac{1}{2}\text{area}(B)$, *provided* we show that f is continuous.

Why is f continuous? Fix $u, v \in [-M, M]$. Then $|f(u) - f(v)|$ is the area of $B \cap \{(x, y) : u \leq x \leq v\}$. Since B is contained in $[-M, M]^2$, we see that $|f(x) - f(y)| \leq 2M|x - y|$. Therefore, f is continuous (given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2M}$ to check the $\varepsilon - \delta$ definition of continuity). ■